Discrete controller Design

Pole-Placement

Digital control

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Pole-Placement Control – Analytical

- The response of a system is determined by the positions of its closed-loop poles. Thus, by placing the poles at the required points we should be able to control the response of a system.

Given the pole positions of a system, (9.3) gives the required transfer function of the controller as

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)}.$$

T(z) is the required transfer function, which is normally in the form of a polynomial.
The denominator of $T(z)$ is constructed from the positions of the required roots.
The numerator polynomial can then be selected to satisfy certain criteria in the system.

Pole-Placement Control: example

Example 9.3

The open-loop transfer function of a system together with a zero-order hold is given by

$$HG(z) = \frac{0.03(z + 0.75)}{z^2 - 1.5z + 0.5}.$$

Design a digital controller so that the closed-loop system will have $\zeta = 0.6$ and $w_d = 3$ rad/s. The steady-state error to a step input should be zero. Also, the steady-state error to a ramp input should be 0.2. Assume that $T = 0.2$ s.

Solution

The roots of a second-order system are given by

$$z_{1,2} = e^{-\zeta\omega_n T \pm j\omega_n T\sqrt{1-\zeta^2}} = e^{-\zeta w_n T}(\cos\omega_n T\sqrt{1-\zeta^2} \pm j\sin\omega_n T\sqrt{1-\zeta^2}).$$

Pole-Placement Control: example

Thus, the required pole positions are

$$z_{1,2} = e^{-0.6\times3.75\times0.2}(\cos(0.2\times3)\pm j\sin(0.2\times3)) = 0.526 \pm j0.360.$$

The required controller then has the transfer function

$$T(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots}{(z - 0.526 + j0.360)(z - 0.526 - j0.360)}$$

which gives

$$T(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots}{1 - 1.052 z^{-1} + 0.405 z^{-2}}. \tag{9.6}$$

We now have to determine the parameters of the numerator polynomial. To ensure realizability, $b_0 = 0$ and the numerator must only have the b_1 and $b_2$terms. Equation (9.6) then becomes

$$T(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 - 1.052 z^{-1} + 0.405 z^{-2}}. \tag{9.7}$$

The other parameters can be determined from the steady-state requirements.

The steady-state error is given by

$$E(z) = R(z)[1 - T(z)].$$

Pole-Placement Control: example

For a unit step input, the steady-state error can be determined from the final value theorem, i.e.

$$E_{ss} = \lim_{z \to 1} \frac{z-1}{z} \frac{z}{z-1} [v]$$

or

$$E_{ss} = 1 - T(1). \tag{9.8}$$

From (9.8), for a zero steady-state error to a step input,

$$T(1) = 1$$

From (9.7), we have

$$T(1) = \frac{b_1 + b_2}{0.353} = 1$$

or

$$b_1 + b_2 = 0.353, \tag{9.9}$$

and

$$T(z) = \frac{b_1 z + b_2}{z^2 - 1.052z + 0.405}. \tag{9.10}$$

Pole-Placement Control: example

If K_v is the system velocity constant, for a steady-state error to a ramp input we can write

$$E_{ss} = \lim_{z \to 1} \frac{(z-1)}{z} \frac{Tz}{(z-1)^2}[1 - T(z)] = \frac{1}{K_v}$$

or, using L'Hospital's rule,

$$\left.\frac{dT}{dz}\right|_{z=1} = -\frac{1}{K_v T}.$$

Pole-Placement Control: example

Thus from (9.10),

$$\left.\frac{dT}{dz}\right|_{z=1} = \frac{b_1(z^2 - 1.052z + 0.405) - (b_1 z + b_2)(2z - 1.052)}{(z^2 - 1.052z + 0.405)^2} = -\frac{1}{K_v T} = -\frac{0.2}{0.2} = -1,$$

giving

$$\frac{0.353b_1 - (b_1 + b_2)0.948}{0.353^2} = -1$$

or

$$0.595b_1 + 0.948b_2 = 0.124, \tag{9.11}$$

From (9.9) and (9.11) we obtain,

$$b_1 = 0.596 \text{ and } b_2 = -0.243.$$

Equation (9.10) then becomes

$$T(z) = \frac{0.596z - 0.243}{z^2 - 1.052z + 0.405}. \tag{9.12}$$

Pole-Placement Control: example

Equation (9.12) is the required transfer function. We can substitute in Equation (9.3) to find the transfer function of the controller:

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)} = \frac{z^2 - 1.5z + 0.5}{0.03(z + 0.75)} \frac{T(z)}{1 - T(z)}$$

or,

$$D(z) = \frac{z^2 - 1.5z + 0.5}{0.03(z + 0.75)} \frac{0.596z - 0.243}{z^2 - 1.648z + 0.648}$$

which can be written as

$$D(z) = \frac{0.596z^3 - 1.137z^2 + 0.662z - 0.121}{0.03z^3 - 0.027z^2 - 0.018z + 0.015} \tag{9.13}$$

The step response of the system with the controller is shown in Figure 9.10.

Pole-Placement Control: example

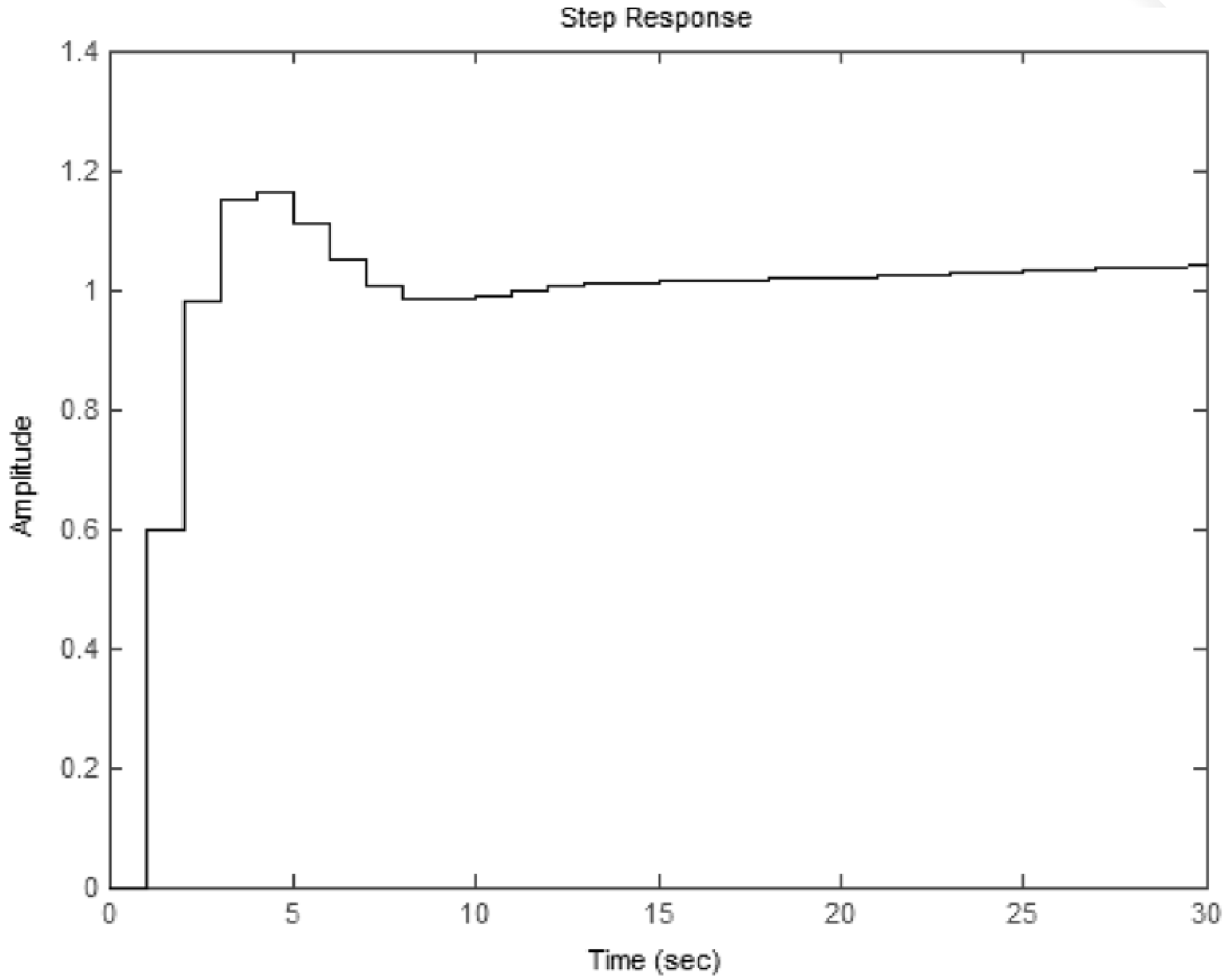

Figure 9.10 Step response of the system

Pole placement example

Example 9.6

The block diagram of a system is as shown in Figure 9.19. It is required to design a controller for this system with percent overshoot (PO) less than 17 % and settling time $t_s \leq 10$ s. Assume that $T = 0.1$ s.

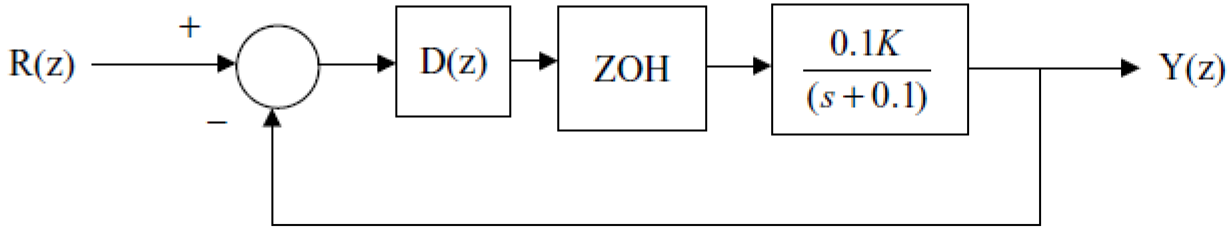

Figure 9.19 Block diagram for Example 9.6

Pole placement example

Solution

The damping ratio, natural frequency and hence the required root positions can be determined as follows:

$$\text{For } PO < 17\%, \quad \zeta \geq 0.5.$$

$$\text{For } t_s \leq 10, \quad \zeta\omega_n \geq \frac{4.6}{t_s} \quad \text{or} \quad \omega_n \geq 0.92\,\text{rad/s}.$$

Hence, the required pole positions are found to be

$$z_{1,2} = e^{-\zeta\omega_n T}\left(\cos\omega_n T\sqrt{1-\zeta^2} + j\sin\omega_n T\sqrt{1-\zeta^2}\right)$$

or

$$z_{1,2} = 0.441 \pm j0.451.$$

The z-transform of the plant, together with the zero-order hold, is given by

$$G(z) = \frac{z-1}{z} Z\left[\frac{0.1K}{s^2(s+0.1)}\right] = \frac{0.00484K(z+0.9672)}{(z-1)(z-0.9048)}.$$

Pole placement example

It is clear from the figure that the root locus will not pass through the marked point by simply changing the d.c. gain. We can design a compensator as in Example 9.5 such that the locus passes through the required point, i.e.

$$D(Z) = \frac{z-n}{z-p}.$$

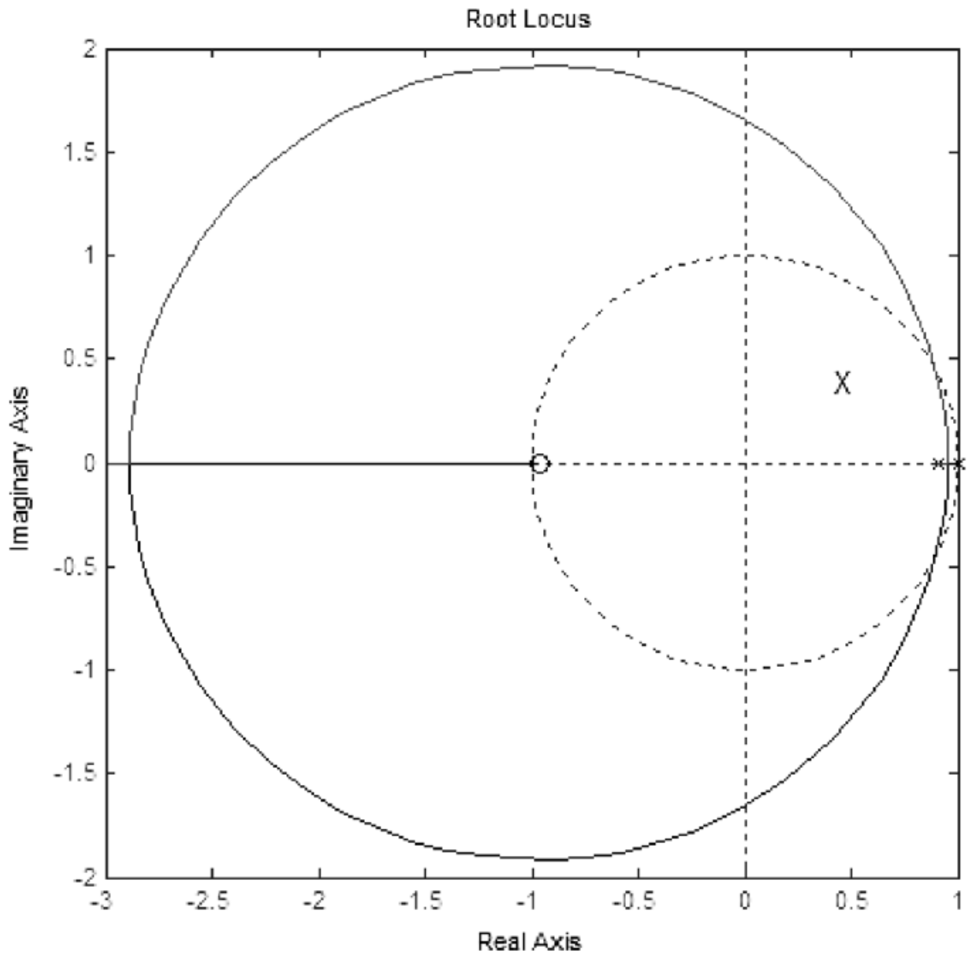

Figure 9.20 Root locus of uncompensated system

Pole placement example

The angle of $G(z)$ at the required point is

$$\angle G(z) = \angle 0.441 + j0.451 + 0.9672 - \angle(0.441 + j0.451 - 1) - \angle(0.441 + j0.451 - 0.9048)$$

or

$$\angle G(z) = \tan^{-1}\frac{0.451}{1.4082} - \tan^{-1}\frac{0.451}{-0.559} - \tan^{-1}\frac{0.451}{-0.4638} = -259^\circ.$$

Since the sum of the angles at a point in root locus must be a multiple of -180°, the compensator must introduce an angle of $-180 - (-259) = 79^\circ$. The required angle can be obtained using a compensator with a transfer function, and the angle introduced by the compensator is

$$\angle D(Z) = \angle(0.441 + j0.451 - n) - \angle(0.441 + j0.451 - p) = 79^\circ$$

or

$$\tan^{-1}\frac{0.451}{0.441 - n} - \tan^{-1}\frac{0.451}{0.441 - p} = 79^\circ.$$

If we choose $n = 0.6$, then

$$109^\circ - \tan^{-1}\frac{0.451}{0.441 - p} = 79^\circ$$

or

$$p = -0.340.$$

Pole placement example

The transfer function of the compensator is thus

$$D(z) = \frac{z - 0.6}{z + 0.340}.$$

Figure 9.21 shows the root locus of the compensated system. Clearly the locus passes through the required point. The d.c. gain at this point is $K = 123.9$.

The time response of the compensated system is shown in Figure 9.22.

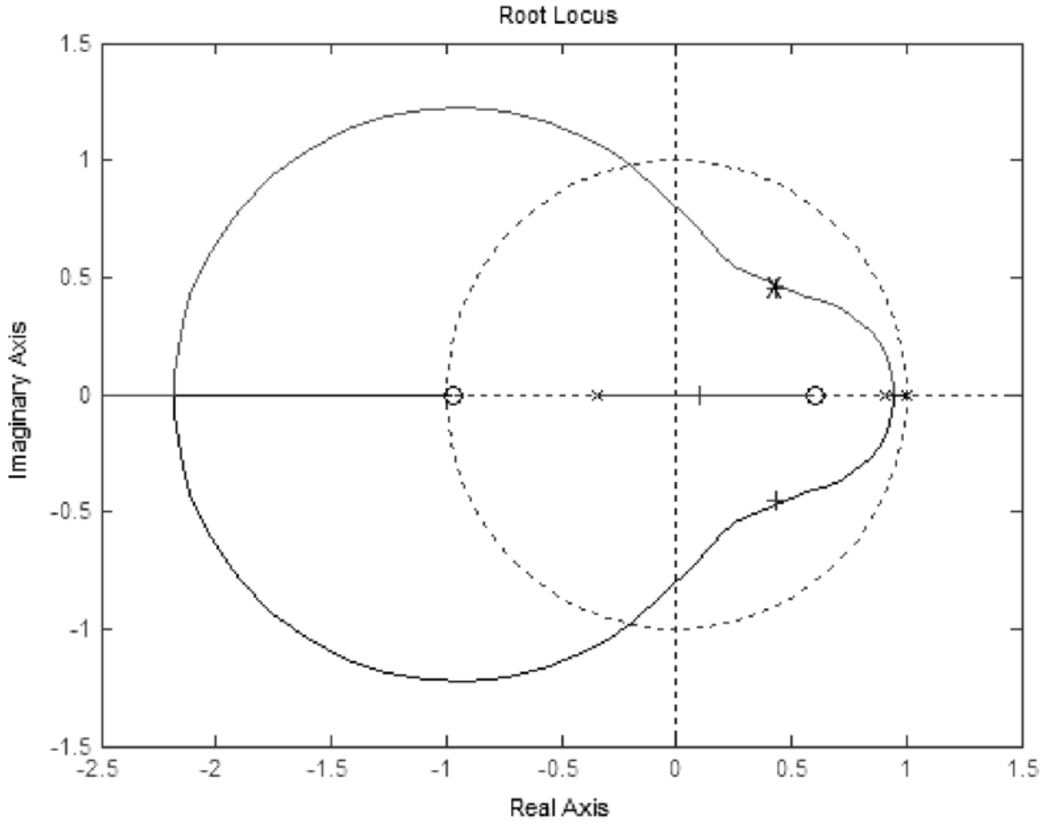

Figure 9.21 Root locus of the compensated system

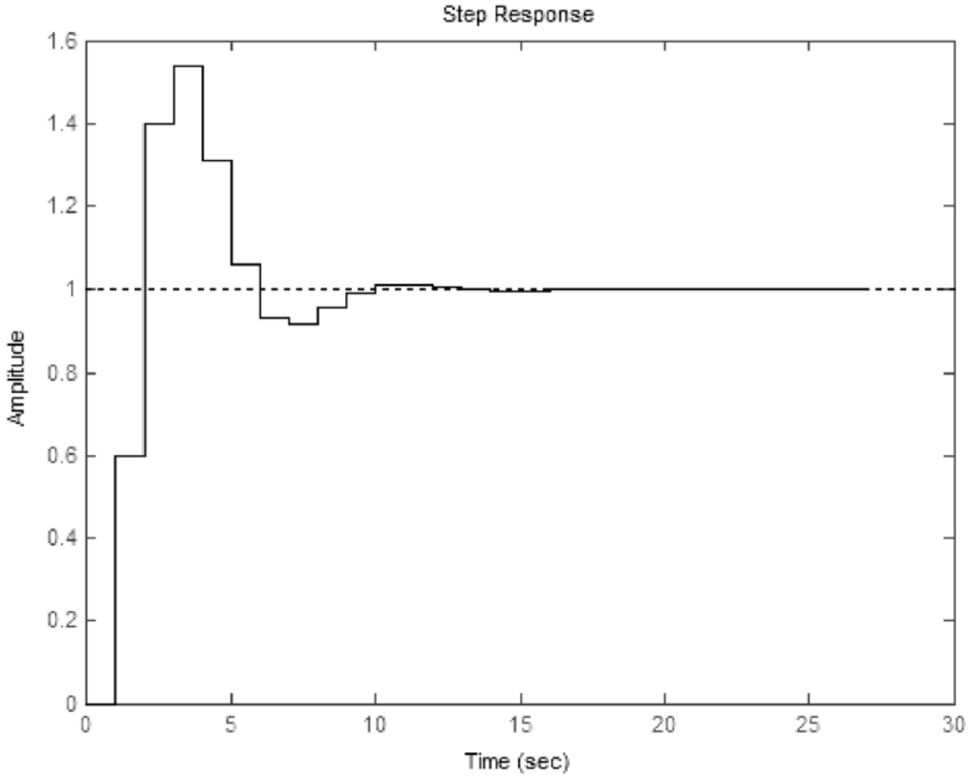

Figure 9.22 Time response of the system